Indian Statistical Institute, Bangalore

B. Math (Hons.) Third Year

First Semester - Optimization

Semestral Exam — Date: November 22, 2018

Maximum marks: 50 — Duration: 3 hours

Section I: Answer any four and each question carries 6 marks.

1. Prove QR-decomposition for full column rank matrices and prove the decomposition is unique if R is required to have positive entries on the diagonal.
2. If $A = UDV^t$ is the SVD, prove that range of A and null space of A are spanned by $\{u_i \mid i \leq r\}$ and $\{v_i \mid i > r\}$ respectively for some r.
3. Let $T \geq S \geq 0$ and T be irreducible and $T - S$ is non-negative. Prove that $\mathrm{Spr}(T) \geq \mathrm{Spr}(S)$ and the equality occurs only if $T = S$.
4. Solve
$$\begin{array}{cc} \text{Maximize} & 3x_1 + 2x_2 + x_3 \\ \text{subject to} & 2x_1 + x_2 + x_3 \leq 150 \\ & 2x_1 + 2x_2 + 8x_3 \leq 200 \\ & 2x_1 + 3x_2 + x_3 \leq 320 \\ \text{with} & x \geq 0. \end{array}$$
5. Describe and justify a method to avoid anticycling in LP.
6. Each of two players shows one or two fingers (simultaneously) and C pays to R an amount equal to the total number of fingers shown, while R pays to C an amount equal to the product of the numbers of fingers shown. Construct the game matrix (the entries will be the net gain of R), and find the value of the game and the optimal strategies.

Section II: Answer any two and each question carries 13 marks.

1. (a) Let s and t be the largest and least singular values of A. Show that $||A|| = s$ and $\mathrm{mini}_{||x||=1}||Ax|| = t$ or 0(*Marks: 7*).

   (b) Use Perron-Frobenius theory to compute $\lim_{n\to\infty} A^n$, for $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$.

2. (a) If $A > 0$ with spectral radius 1, prove that $Ax = x$ implies $A|x| = |x|$ where $|x| = (|x_i|)$ (*Marks: 6*).

   (b) State and prove a sufficient condition for a LP to be unbounded (*Marks 4*).

   (c) Find the dual of

$$\begin{array}{cc} \text{Maximize} & e^t x + f^t y \\ \text{subject to} & Ax + By = g \\ & Cx + Dy \leq h \\ \text{with} & x \geq 0. \end{array}$$

3. (a) State and prove MinMax Theorem (*Marks: 7*).

   (b) Find a necessary and sufficient condition for the game $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ to be nonstrictly determined and solve the game when it is nonstrictly determined.